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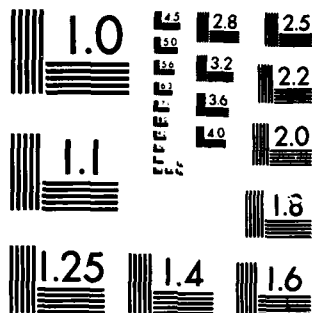
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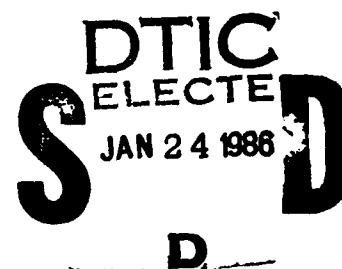


IRREDUNDANCY IN MULTIPLE INTERVAL REPRESENTATIONS

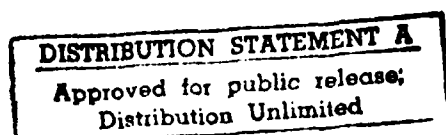
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Technical Report No. 411  
ONR Technical Report No. 84-1  
September, 1984



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## Irredundancy in Multiple Interval Representations

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### ABSTRACT

In a multiple interval intersection representation of a graph it is *required* that at least one interval from each of a pair of adjacent vertices intersect. It is *permitted* for there to be several such intersections, even though these additional intersections are *superfluous* or *redundant*. By disallowing such redundancies one arrives at the concept of an *irredundant* multiple interval representation. We show that these irredundant representations can be much more *inefficient* than representations which allow redundancies. Finally, we show that even when some redundancy is permitted, the inefficiency remains.

September 1984

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# Irredundancy in Multiple Interval Representations

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## 1. Introduction

Interval graphs are well known: they are the graphs to whose vertices one can assign real intervals such that vertices are adjacent if and only if their intervals intersect. Recently, multiple interval graphs have been studied [1-7].

A  $t$ -interval is the union of  $t$  real closed intervals and a  $t$ -interval graph is the intersection graph of  $t$ -intervals. Stated differently, a graph is a  $t$ -interval graph if to each vertex of the graph one can assign (up to)  $t$  real intervals so that two vertices are adjacent if and only if some interval assigned to one vertex intersects an interval assigned to the second.

We introduce some notation. Let  $tI$  denote the family of all  $t$ -intervals:

$$tI = \{[a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_t, b_t] : a_i < b_i, 1 \leq i \leq t\}.$$

Notice that the intervals need not be distinct nor disjoint. Thus  $tI \subset (t+1)I$  for all  $t$ . We write  $v \sim w$  when vertices  $v$  and  $w$  are adjacent. Then a  $t$ -interval graph is one with a  $t$ -interval representation:

$$f: V(G) \rightarrow tI$$

where  $v \sim w$  iff  $f(v) \cap f(w) \neq \emptyset$ .

Every graph is a  $t$ -interval graph for  $t$  sufficiently large. One defines the interval number of a graph  $G$ , denoted  $i(G)$ , to be the least  $t$  for which  $G$  is a  $t$ -interval graph. Interval graphs are precisely those graphs,  $G$ , with  $i(G)=1$ .

Given a  $t$ -interval representation of a graph  $G$ , it is possible for several intervals assigned to one vertex to meet several intervals assigned to one of its

neighbors. Only one such intersection is required; the others are permitted but are, in some sense, superfluous or redundant. The aim of this paper is to study the effect redundancy has on multiple interval representations of graphs.

A  $t$ -interval representation,  $f: V(G) \rightarrow tI$ , is called *irredundant* if for all  $v, w$  with  $v \sim w$ ,  $f(v) \cap f(w)$  is an interval. The *irredundant interval number* of  $G$ , denoted  $i_0(G)$ , is the least integer  $t$  such that  $G$  has an irredundant  $t$ -interval representation. The subscript "0" indicates that no redundancy is allowed.

Clearly  $i_0(G) \geq i(G)$  since irredundant representations are themselves representations. Small examples suggest that  $i_0(G) = i(G)$  and it would be reasonable to conjecture this equality always holds because representing edges with more than one interval-interval intersection would "waste" intervals that could be used to represent other edges. However, we show in the next section how to construct graphs with  $i_0(G) = i(G) + 1$ . One is then led to ask how different the parameters  $i$  and  $i_0$  can be. After several technical results in section 3 we show, in section 4, that  $i_0(G)$  can be arbitrary while  $i(G) = 2$ .

## 2. A First Example

If  $i(G) = 1$  then it is immediate that  $i_0(G) = 1$ . We show in this section:

**Theorem 1.** For every integer  $t > 1$  there exists a graph  $G$  with  $i(G) = t$  and  $i_0(G) = t + 1$ .

To prove this we use the concept of tightness introduced in [7].

A graph  $G$  is called  $t$ -tight provided that for every  $t$ -interval representation,  $f: V(G) \rightarrow tI$ , one has  $\bigcup_{v \in V(G)} f(v)$  is an interval. In other words, in  $G$ 's  $t$ -interval representation there can be no "gaps". If  $G$  is  $t$ -tight, one readily verifies that  $i(G) = t$ .

**Lemma 1 [7].** The complete bipartite graph  $K_{2t+1, 2t-1}$  is  $t$ -tight.

Also, it can be shown that any vertex can appear as the "first" interval (ordered by left endpoint) in a  $t$ -interval representation of  $K_{2t+1, 2t-1}[\#]$

Proof of Theorem 1. Let  $t > 1$  be an integer. Construct a graph  $G$  consisting of  $2t$  disjoint copies of  $K_{2t+1, 2t-1}$ , each with a distinguished vertex, plus two additional vertices,  $x$  and  $y$ . Join  $x$  to  $y$  with an edge, and join both of them to each of the distinguished vertices in the  $2t$  copies of  $K_{2t+1, 2t-1}$ . See Figure 1.

One now checks that  $i(G) = t$  by consulting Figure 2. Since  $K_{2t+1, 2t-1}$  is  $t$ -tight, its intervals must cover an unbroken portion of the real line. Thus in any  $t$ -interval representation of  $G$  the  $K_{2t+1, 2t-1}$ 's cover  $2t$  intervals on the real line. In order for  $x$  to meet the appropriate distinguished vertices we must put  $t$  intervals in the gaps between the first and second, third and fourth, etc. copies of  $K_{2t+1, 2t-1}$ . Likewise for  $y$ . We now see that edge  $xy$  is represented " $t$  times"; no irredundant  $t$ -interval representation of  $G$  is possible. Hence  $i_0(G) \geq t+1$ . It is easy to give a simple construction to show that  $i_0(G) = t+1$ .

Thus the parameters  $i_0$  and  $i$  are not equal, yet intuitively one might expect them to be close. However, this is not even remotely correct. In the next section we present some technical material we use in section 4 where we show that  $i_0(G)$  can be arbitrarily large while  $i(G) = 2$ .

### 3. Two Lemmas

In this section we present two results which we use repeatedly to prove the main theorem. The first concerns multiple interval representations of complete graphs and the second is a "probabilistic" pigeon hole result.

Let  $f: V(G) \rightarrow tI$  be a  $t$ -interval representations of a graph  $G$  and let  $x$  be a point on the real line. The *depth* of the representations at  $x$  is the number of vertices assigned to intervals containing  $x$ :

$$\text{depth}(x) = |\{v \in V(G): x \in f(v)\}|$$

The *depth* of the representation is the largest such number:

$$\text{depth}(f) = \sup_{x \in \mathbb{N}} \text{depth}(x)$$

The depth of a 1-interval representation of a complete graph,  $K_n$  is  $n$ ; this is Helly's theorem. For multiple interval representations we have:

Lemma 2. The depth of a  $t$ -interval representation of a complete graph  $K_n$  exceeds  $\frac{n}{2t}$ .

Proof. Let  $f: (V(K_n)) \rightarrow tI$  be a  $t$ -interval representation for  $K_n$  and let  $r = \text{depth}(f)$ . Without loss of generality we may assume that all end points of all the intervals in  $f$  are distinct. Each left end point of an interval in  $f$  is contained in at most  $r-1$  other intervals. Since there are at most  $nt$  intervals in  $f$  and for each edge of  $K_n$  there is a left end point of an interval contained in another interval, we have:

$$\begin{aligned} |E(K_n)| &\leq (r-1)nt \\ \frac{1}{2}n(n-1) &\leq (r-1)nt \end{aligned}$$

hence:

$$r \geq \frac{n-1}{2t} + 1 > \frac{n}{2t}.$$

Next we present a "probabilistic" pigeon hole result:

Lemma 3. Let  $0 < \varepsilon < 1$  and  $\delta = \frac{\varepsilon}{2}$ . Let  $C(1), C(2), \dots, C(p)$  be disjoint finite sets each of cardinality  $q$ , and let  $C(*)$  denote their union. Suppose  $S$  is a subset of  $C(*)$  with  $|S| \geq \varepsilon |C(*)| = \varepsilon pq$ . Then the number of indices  $k$  for which:

$$|S \cap C(k)| \geq \delta |C(k)| = \delta q \tag{1}$$

is at least  $\delta p$ . In symbols:

$$|\{k: |S \cap C(k)| \geq \delta q\}| \geq \delta p.$$

In "pigeon language", the  $C(k)$  represent the equal capacity pigeon holes and  $S$  represents the pigeons. If the coop is moderately full (" $\epsilon\%$ ") then a fair number of holes (" $\delta\%$ ") are fairly full (" $\delta\%$ ").

Proof of the lemma. Suppose fewer than  $\delta p$  of the  $C(k)$  satisfy (1). That means that up to  $\delta p - 1$  of the  $C(k)$  can have "a lot" (but at most  $q = |C(k)|$ ) of their elements in  $S$  while the remaining  $p - \delta p + 1$   $C(k)$ 's can have at most  $\delta q - 1$  elements in  $S$ . Thus,

$$\begin{aligned} |S| &\leq (\delta p - 1)q + (p - \delta p + 1)(\delta q - 1) = \\ &= 2\delta pq - (1 - \delta)p - (1 - \delta)q - \delta^2 pq - 1 < \\ &< 2\delta pq = \epsilon pq. \end{aligned}$$

But  $|S| \geq \epsilon pq$  by hypothesis.

#### 4. Main Result

We now present and prove the principal result of this paper.

Theorem 2. For every positive integer  $t$  there exists a graph  $G$  with  $i(G) = 2$  and  $i_0(G) > t$ .

Proof. We explicitly construct the graphs  $G$ . For positive integers  $n, m, q$  define a graph  $G(n, m; q)$  as follows. The vertices of  $G(n, m; q)$  are triples of integers  $(i, j; k)$  with  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  and  $1 \leq k \leq q$ , i.e.,

$$V(G(n, m; q)) = \{1, \dots, n\} \times \{1, \dots, m\} \times \{1, \dots, q\}$$

We put  $(i, j; k) \sim (i', j'; k')$  if and only if  $i = i'$  or  $j = j'$ . (Notice that the entry in the third coordinate does not matter.) See figure 3.

If  $n$  and  $m$  are both greater than 1, then  $(1, 1; 1) - (1, 2; 1) - (2, 2; 1) - (2, 1; 1) - (1, 1; 1)$  is an induced 4-cycle and hence  $G(n, m; q)$  is not an interval graph. Therefore  $i(G(n, m; q)) \geq 2$ . We now show that its interval number is exactly 2 by explicitly constructing a representation.

Define  $f: V(G(n, m; q)) \rightarrow 2I$  by

$$f(i, j; k) = [i - \frac{1}{4}, i + \frac{1}{4}] \cup [-j + \frac{1}{4}, -j - \frac{1}{4}].$$

It is easy to check that  $f$  is a 2-interval representation of  $G(n, m; q)$ . (Notice that  $f$  is highly redundant. Every edge of the form  $(i, j; k)(i, j; k')$  is represented twice.)

Pick an integer  $t > 1$ . We show that for suitable  $n, m_0, q_0$  we have  $i_0(G(n, m_0; q_0)) > t$ . Here suitable with entail  $t \ll n \ll m_0 \ll q_0$ . In particular we take

$$n = 12t$$

$$m_0 = 2(4t)^n$$

$$q_0 = 2(4t)^{n+1}.$$

Let  $G_0 = G(n, m_0; q_0)$ . Observe that for all  $i$  and  $j$  the set of vertices  $C_0(i, j) = \{(i, j; k) : 1 \leq k \leq q_0\}$  induces a clique containing  $q_0$  vertices. Also  $C_0(i, *) = C_0(i, 1) \cup \dots \cup C_0(i, m_0)$  induces a clique containing  $m_0 q_0$  vertices and  $C_0(*, j) = C_0(1, j) \cup \dots \cup C_0(n, j)$  induces a clique containing  $n q_0$  vertices.

Suppose  $i_0(G_0) \leq t$ . Fix an irredundant  $t$ -interval representation  $f$  for  $G_0$ . Let  $\varepsilon = \frac{1}{2t}$  and let  $\delta = \frac{\varepsilon}{2}$ . Consider the clique  $C_0(1, *)$ . By lemma 2 its representation in  $f$  has depth at least (in fact exceeding)  $\frac{|C_0(1, *)|}{2t} = \varepsilon m_0 q_0$ . Thus some point  $x_1$  on the real line is contained in intervals for at least  $\varepsilon m_0 q_0$  vertices of  $C_0(1, *)$ . We call the intervals assigned to vertices of  $C_0(1, *)$  which contain the point  $x_1$  a *stack* and denote the collection  $S(x_1)$ . If a vertex in  $C_0(1, *)$  has an interval in the stack  $S(x_1)$  we call that interval *primary* and the remaining  $t-1$  intervals *secondary*. We now apply lemma 3. Observe that  $\varepsilon(m_0 q_0)$  of the vertices in  $C_0(1, *)$  have a (primary) interval in  $S(x_1)$ . Therefore at least  $\delta m_0$  of the cliques  $C_0(1, j)$  have at least  $\delta q_0$  vertices with (primary) intervals in  $S(x_1)$ . We may assume, after appropriate relabeling, that this is true for cliques

$C_0(1,1), \dots, C_0(1, \delta m_0)$  and that vertices  $(1,j;k)$  with  $1 \leq j \leq \delta m_0$  and  $1 \leq k \leq \delta q_0$  have primary intervals in  $S(x_1)$ , i.e.  $x_1 \in f(1,j;k)$ .

We now restrict our attention to an induced subgraph  $G_1$  of  $G_0$ . Put  $m_1 = \delta m_0$ ,  $q_1 = \delta q_0$  and  $G_1 = G(n, m_1; q_1)$ . (Notice that we do not alter  $f$  but restrict it to  $V(G_1)$ .) By analogy we define  $C_1(i,j)$ ,  $C_1(i,*)$  and  $C_1(*,j)$ . Notice that, by our analysis, all vertices in  $C_1(1,*)$  have a (primary) interval contained in  $S(x_1)$ . We now repeat the above argument for  $C_1(2,*)$ : There exists a point  $x_2$  on the real line and, after suitable relabeling, for  $1 \leq j \leq \delta m_1$  and  $1 \leq k \leq \delta q_1$  we have  $x_2 \in f(2,j;k)$ . We call the intervals assigned to vertices in  $C_1(2,*)$  which contain  $x_2$  stack  $S(x_2)$ . The designations "primary" and "secondary" are clear in this context.

Put  $m_2 = \delta m_1$  and  $q_2 = \delta q_1$ . We let  $G_2 = G(n, m_2; q_2)$  and note that it has the property that all vertices in  $C_2(1,*)$  and  $C_2(2,*)$  have primary intervals in stacks  $S(x_1)$  and  $S(x_2)$  respectively.

We now continue to define  $G_3$ ,  $G_4$ , etc. After  $n$  iterations we have  $m = m_n = \delta^n m_0 = 2$  and  $q = q_n = \delta^n q_0 = 8t$ . Put  $G = G_n = G(n, m; q)$ . For all  $i = 1, \dots, n$  we know that the  $mq$  vertices in  $C(i,*) = C_n(i,*)$  have primary intervals containing the point  $x_i$ . Thus for all  $(i,j;k) \in V(G)$  we have  $x_i \in f(i,j;k)$ .

Suppose vertices  $v$  and  $w$  are in clique  $C(i,*)$ . Their primary intervals intersect:  $x_i \in f(v) \cap f(w)$ . Since  $f$  is irredundant, their  $2(t-1)$  secondary intervals must be disjoint.

Without loss of generality, we may assume  $x_1 < x_2 < \dots < x_n$ . We now claim that no primary interval containing  $x_i$  can contain  $x_{i'}$  for any  $i' \neq i$ ; otherwise there would be a vertex  $v$  adjacent to all vertices in both  $C(i,*)$  and  $C(i',*)$ . This implies that  $v \in C(i,*)$  since if  $v = (i'', j; k)$  with  $i'' \neq i$  then (since  $m = 2$ ) there exists  $j' \neq j$  and  $(i'', j', k)$  is not adjacent to  $(i, j'; k) \in C(i,*)$ . The same reasoning shows  $v \in C(i',*)$  but  $C(i,*) \cap C(i',*) = \emptyset$  and the claim follows. We now may

conclude that if  $|i-i'| > 1$  then the primary intervals for  $(i,j;k)$  and  $(i',j';k')$  are disjoint. Primary intervals from non-consecutive stacks cannot meet.

Finally, consider the clique  $C(*,1)$ . It has  $nq$  vertices. By the usual argument there exists a point  $y$  on the real line containing  $\varepsilon nq$  intervals from  $C(*,1)$ . Moreover, at least  $\delta n$  of the cliques  $C(i,1)$  have at least  $\delta q$  vertices in the stack  $S(y)$ . Note that  $\delta n = 3$ . Thus there exist indices  $i$  and  $i'$  with  $|i-i'| > 1$  so that cliques  $C(i,1)$  and  $C(i',1)$  each have at least  $\delta q = 2$  intervals in stack  $S(y)$ . Since  $\delta q = 2 > 1$ , these intervals must be primary, since secondary intervals belonging to a pair of vertices in a  $C(i,j)$  cannot intersect. However, this is a contradiction because these primary intervals belong to the non-consecutive stacks  $S(i)$  and  $S(i')$  and are therefore disjoint. Thus  $i_0(G_0) > t$ , yet  $i(G_0) = 2$ .

This result is best possible since, as we noted earlier, if  $i(G) = 1$  then  $i_0(G) = 1$ .

The relationship between the graph  $G(n,m;q)$  and  $G(n,m_0;q_0)$  is worth noting now as its own result:

Lemma 4. Given positive integers  $t, n, m_0$  and  $q_0$ , there exist integers  $m$  and  $q$  such that the following holds: If  $G = G(n,m;q)$  and  $f$  is a  $t$ -interval representation of  $G$ , then there exists an induced subgraph  $G'$  of  $G$ , isomorphic to  $G(n,m_0;q_0)$ , and distinct points  $x_1, \dots, x_n \in \mathbb{R}$  such that  $x_i \in f(i,j;k)$  whenever  $(i,j;k) \in V(G')$ .

## 5. Remarks on the Large Cliques

One of the most striking features of the  $G(n,m;q)$  graphs in the above proof is their very large cliques. This feature, to an extent, is inescapable, as shown below. We begin with a lemma.

Lemma 5. Let  $f: V(G) \rightarrow tI$  be a  $t$ -interval representation of  $G$ . If  $\text{depth}(f) = 2$

then  $i_0(G) \leq t$ .

Proof. If  $f$  is irredundant we have nothing to prove. Otherwise we remove "superfluous" intersections as follows:

There are essentially only two ways for a pair intervals to meet in a representation with depth-2. Either one interval is contained in the second, or else each interval contains exactly one end point of the other. If such an intersection is superfluous, we modify  $f$  in the first case by deleting the smaller interval and in the second case by "sliding" the overlapping intervals apart. Repeating this process for all "superfluous" intersections gives an irredundant representation.■

Let  $\omega(G)$  denote the number of vertices in  $G$ 's largest clique. Although one cannot give a bound for  $i_0$  in terms of  $i$  alone, one can give a bound when  $\omega$  is known.

Theorem 3.  $i_0(G) \leq (\omega(G)-1)i(G)$ .

Proof. Let  $f$  be an  $i(G)$ -interval representation of  $G$ . Clearly  $\text{depth}(f) \leq \omega(G)$ . One can imagine a drawing of such a representation as occupying  $\omega(G)$  "layers" in which each interval lies in one of the layers and meets no other interval in its layer. Now all edges represented by intersections between a given pair of layers can be represented in a depth-2 fashion by "recopying" the two layers in an unused portion of the line. We do this in all  $\binom{\omega(G)}{2}$  possible ways. One checks that each layer is recopied  $\omega(G)-1$  times resulting in a  $(\omega(G)-1)i(G)$ -representation with depth-2. By lemma 5,  $i_0(G) \leq (\omega(G)-1)i(G)$ .■

Corollary. For triangle free graphs,  $i_0(G) = i(G)$ .■

## 6. $r$ -Redundancy

Until now we have been discussing redundancy in an all-or-nothing manner. In section 4 we saw that the graphs  $G(n, m; q)$  do not have "efficient" irredundant representations. However, if we allow (up to) two intervals from a vertex to meet intervals from a neighbor, we can form an efficient ( $i=2$ ) multiple interval representation for  $G(n, m; q)$ . One is therefore led to ask: if we allow a "little" redundancy do we still adversely affect efficiency?

When we defined irredundant  $t$ -interval representations we placed the restriction that  $f(v) \cap f(w)$  may consist of at most one connected component (i.e. is empty or is an interval). One way to allow a "little" redundancy is to place a fixed upper bound on the number of components in  $f(v) \cap f(w)$ . We take a different but qualitatively equivalent approach:

For  $r \geq 0$  we say that a  $t$ -interval representation of a graph  $G$  is  $r$ -redundant if, for all  $v, w \in V(G)$ , at most  $r+1$  intervals assigned to  $v$  meet intervals assigned to  $w$ . [Note that this effectively places an upper bound of  $2r+1$  components in  $f(v) \cap f(w)$ .] This notation was chosen so that 0-redundant and irredundant would be synonymous. We denote by  $i_r(G)$  the least  $t$  for which  $G$  has an  $r$ -redundant  $t$ -interval representation.

The following facts are obvious:

- (1)  $i(G) \leq i_{r+1}(G) \leq i_r(G)$ , and
- (2) if  $i(G) \leq r+1$  then  $i(G) = i_r(G)$ .

Is there an  $r$  for which  $i_r(G) = i(G)$  always holds? Clearly not by the examples in section 2; those graphs have arbitrarily high redundancy. Is there an  $r$  for which  $i_r(G)$  and  $i(G)$  are relatively close? The graphs  $G(n, m; q)$  are 2-interval graphs, hence  $i_1(G(n, m; q)) \leq 2$  and therefore do not answer this question. However, by generalizing these graphs we can answer this question in the negative. We show:

Theorem 4. Given integers  $r, t$  with  $r \geq 0$  and  $t \geq r+2$ , there exists a graph  $G$  with  $i(G)=r+2$  and  $i_r(G) > t$ .

As in the proof of Theorem 2, we will make repeated use of Lemmas 2 and 3. The graphs we examine are defined as follows: The graph  $G(n_1, n_2, \dots, n_p; q)$  consists of all  $(p+1)$ -tuples of positive integers  $(i_1, i_2, \dots, i_p; k)$  with  $i_j \leq n_j$  and  $k \leq q$ . We denote  $p$ -tuples by bold letters:  $\mathbf{i} = (i_1, \dots, i_p)$ . Vertices  $(\mathbf{i}; k)$  and  $(\mathbf{i}'; k')$  are adjacent iff  $\mathbf{i} = \mathbf{i}'$  or else  $\mathbf{i}$  and  $\mathbf{i}'$  differ in exactly one coordinate. In case  $p=2$ , this definition is the same as before.

Lemma 6.  $i(G(n_1, n_2, \dots, n_p; q)) \leq p$ .

Proof. For all  $1 \leq i_1 \leq n_1, \dots, 1 \leq i_p \leq n_p$  let  $x(i_1, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p)$  be distinct integers. [There are  $\sum_{j=1}^p \prod_{s \neq j} n_s$  such integers.] Define  $f: V(G(n_1, \dots, n_p; q)) \rightarrow pI$  by  $f(\mathbf{i}; k)$  is the union of the  $p$  intervals of length  $\frac{1}{2}$  centered at the points:  $x(*, i_2, \dots, i_p), x(i_1, *, i_3, \dots, i_p), \dots, x(i_1, \dots, i_{p-1}, *)$ . It is immediate that if  $v \sim w$  then  $f(v) \cap f(w) \neq \emptyset$ . On the other hand, if  $f(\mathbf{i}; k) \cap f(\mathbf{i}'; k') \neq \emptyset$  then the intersection contains one of the integers  $x(i_1, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p)$  implying  $\mathbf{i}$  and  $\mathbf{i}'$  agree in all coordinates except, perhaps, coordinate  $j$ . Therefore  $(\mathbf{i}; k) \sim (\mathbf{i}'; k')$ .

Notice that the representation presented above is  $(p-1)$ -redundant. This representation has certain special properties to which we call attention via the following definitions:

A representation  $f: V(G(n_1, \dots, n_p; q)) \rightarrow pI$  is called *coordinate- $j$  canonical* if there exist points  $x(i_1, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p)$  on the real line so that for all vertices  $(\mathbf{i}; k)$  we have  $x(i_1, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p) \in f(\mathbf{i}; k)$ . The representation is called *canonical* if it is coordinate- $j$  canonical for each  $j$  with  $1 \leq j \leq p$ . Our goal now is to show for "large" graphs  $G(n_1, \dots, n_p; q)$  every  $t$ -interval representa-

tion is canonical on a smaller  $G()$  graph. We do this coordinate by coordinate using arguments similar to those in the proof of Theorem 2.

Lemma 7. Given positive integers  $t, (n_1, \dots, n_{j-1}, \nu_j, n_{j+1}, \dots, n_p), q'$  there exist positive integers  $n_j, q$  such that every  $t$ -interval representation  $f$  of  $G = G(n_1, \dots, n_j, \dots, n_p; q)$  has the following property: there exists an induced subgraph  $G'$  of  $G$  isomorphic to  $G(n_1, \dots, n_{j-1}, \nu_j, n_{j+1}, \dots, n_p; q')$  such that  $f|V(G')$  is coordinate- $j$  canonical.

Proof. Notice that in case  $p=2$  we have exactly Lemma 4. Our proof in this general case follows by direct analogy.

Let  $N = \prod_{s \neq j} n_s$  and let  $n_j = (4t)^N \nu_j$  and  $q = (4t)^N q'$ . Let  $G = G(n_1, \dots, n_j, \dots, n_p; q)$  and  $f: V(G) \rightarrow tI$  be a  $t$ -interval representation. Put  $\varepsilon = \frac{1}{2t}$  and  $\delta = \frac{\varepsilon}{2}$ . We perform the following construction for all values of the indices  $i_1, i_2, \dots, i_{j-1}, i_{j+1}, \dots, i_p$  with  $1 \leq i_s \leq n_s$  with  $s \neq j$ . [Thus our construction is performed  $N$  times.]

Note that the set of all vertices of the form  $(i_1, \dots, i_j, \dots, i_p; k)$  with  $i_j$  fixed and  $1 \leq k \leq q$  forms a clique of  $q$  vertices we denote  $C(i_j)$  and the union  $C(1) \cup \dots \cup C(n_j)$  is a clique which we denote  $C(*)$  containing  $n_j q$  vertices. Since  $f$  is a  $t$ -interval representation, by Lemma 2 it has depth  $\varepsilon n_j q$  on  $C(*)$ . Hence there is a point  $x = x(i_1, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p)$  such that for at least  $\varepsilon n_j q$  of the vertices  $v$  in  $C(*)$  we have  $x \in f(v)$ . As in the proof of Theorem 2, we now apply Lemma 3 to show that at least  $\delta n_j$  of the cliques  $C(1), \dots, C(n_j)$  have at least  $\delta q$  vertices  $v$  with  $x \in f(v)$ . Without loss of generality, we may assume this occurs on cliques  $C(1), \dots, C(\delta n_j)$ . We therefore refocus our attention on the induced subgraph  $G(n_1, \dots, n_{j-1}, \delta n_j, n_{j+1}, \dots, n_p; \delta q)$  and repeat this argument for the next choice of indices  $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_p$  with  $n_j \leftarrow \delta n_j$  and  $q \leftarrow \delta q$ .

When we are done, the parameters  $n_j$  and  $q$  will each have been decreased by a factor of  $\delta^N$  giving  $\nu_j$  and  $q'$  respectively. The resultant graph is easily seen to be coordinate- $j$  canonical under  $f$ . ■

Starting with a graph  $G(\nu_1, \dots, \nu_p; q')$  we can apply this lemma  $p$  times to readily prove the following:

Lemma 8. Given integers  $t, (\nu_1, \dots, \nu_p; q')$  there exist integers  $(n_1, \dots, n_p; q)$  with the following property: Let  $G = G(n_1, \dots, n_p; q)$ . For every  $t$ -interval representation  $f$  of  $G$  there is an induced subgraph  $G'$  of  $G$ , isomorphic to  $G(\nu_1, \dots, \nu_p; q')$ , such that  $f|V(G')$  is canonical. ■

This lemma now provides the basis for proving Theorem 4.

Proof of Theorem 4. Choose parameters  $r \geq 0$  and  $t \geq r+2$  as required in the hypothesis. Let  $p = r+2$ . Choose  $\nu \geq p$  (we could have written  $p$  instead of  $\nu$  everywhere below, but have kept these quantities separate for clarity). Let  $G' = G(\nu, \nu, \dots, \nu; 2)$  ( $p$   $\nu$ 's) and let  $G = G(n_1, n_2, \dots, n_p; q)$  be the graph, whose existence is assured by Lemma 8.

By Lemma 6,  $i(G) \leq p$ . Suppose  $i_r(G) \leq t$ . Let  $f: V(G) \rightarrow I$  be an  $r$ -redundant  $t$ -interval representation. By Lemma 8 there is an induced subgraph isomorphic to  $G'$  on which  $f$  is also canonical.

Hence there exist distinct points on the real line satisfying  $x(i_1, i_2, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p) \in f(i; k)$  for all  $i$  and  $k$ . There are  $p\nu^{p-1}$  such points. Next, define an  $i$ -collection to be the set of points:

$$ls(*, i_2, \dots, i_p), (i_1, *, i_3, \dots, i_p), \dots, (i_1, i_2, \dots, i_{p-1}, *)rs$$

We claim there exists an  $i$ -collection with no two of its points appearing consecutively on the real line. To show this we first remark that there are exactly  $\nu^p$  such collections: one for each  $i$ . Next, we call an  $i$ -collection *ruined* if any pair of its points appear consecutively. Now one checks that for each pair of

consecutive  $x()$  points on the real line there is at most one  $i$ -collection that is ruined by that pair. Hence the number of ruined  $i$ -collections is at most  $p\nu^{p-1}-1$ . Therefore, there are at least  $\nu^p - p\nu^{p-1} + 1 > 0$   $i$ -collections which are not ruined, and we have proved our claim.

We now choose an  $i$  so that the corresponding  $i$ -collection is not ruined. Consider the two vertices  $(i;1)$  and  $(i;2)$ . We know  $x(i_1, \dots, i_{j-1}, *, i_{j+1}, \dots, i_p) \in f(i;k)$  for  $k=1,2$ . Also, we claim that no other  $x()$  point is in  $f(i;k)$ . Otherwise, there exists  $i'$  with  $x' = x(i'_1, \dots, *, \dots, i'_p) \in f(i';k')$  and  $i'$  disagrees with  $i$  in position  $j$  and one other position. Since  $x' \in f(i';k')$  we have  $(i;k) \sim (i';k')$ , a contradiction.

It follows that  $f(i;k)$  for  $k=1,2$  must consist of at least  $p$  disjoint intervals meeting the  $p$  points  $x(i_1, \dots, *, \dots, i_p)$ . Hence at least  $p$  intervals assigned to  $(i;1)$  meet intervals assigned to  $(i;2)$ . But this contradicts the assumption that  $f$  is  $r$ -redundant with  $r=p-2$ . Thus  $i_r(G) > t$ . ■

## 7. Acknowledgements.

The results in sections 1 through 5 have also appeared in the author's doctoral dissertation [4] written at Princeton University under the supervision of D.B. West. The author is also indebted to W.T. Trotter, Jr. for his intuition and encouragement concerning this problem.

This research was supported in part by the U.S. Department of the Navy under Office of Naval Research Contract No. N00014-79-C-0801.

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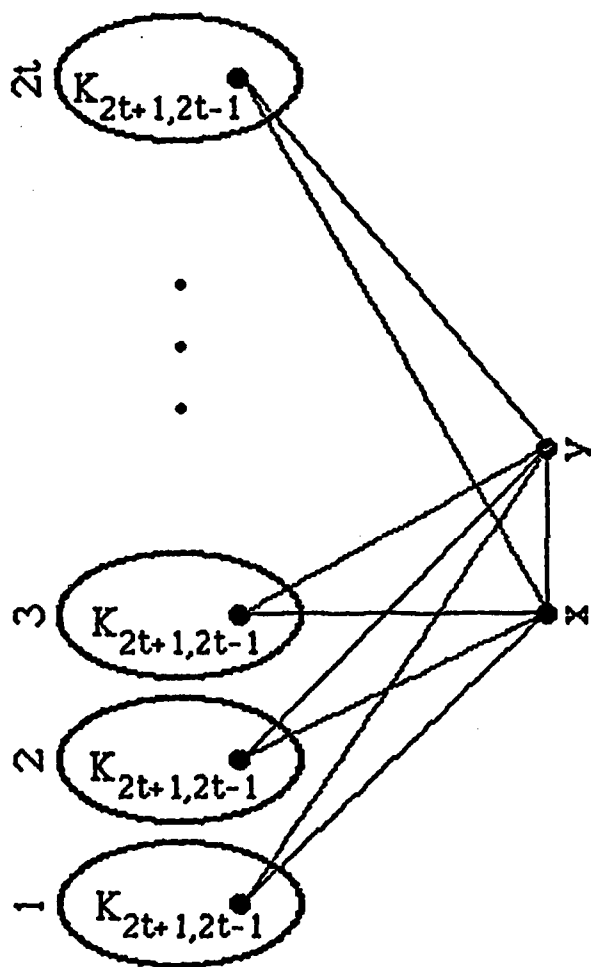


Figure 1.

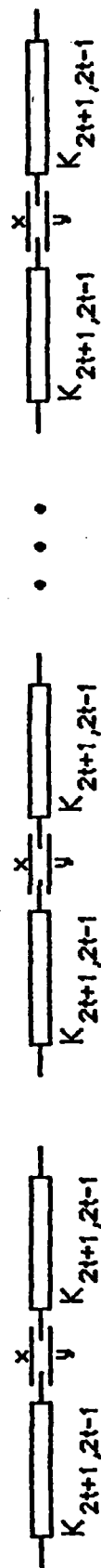


Figure 2

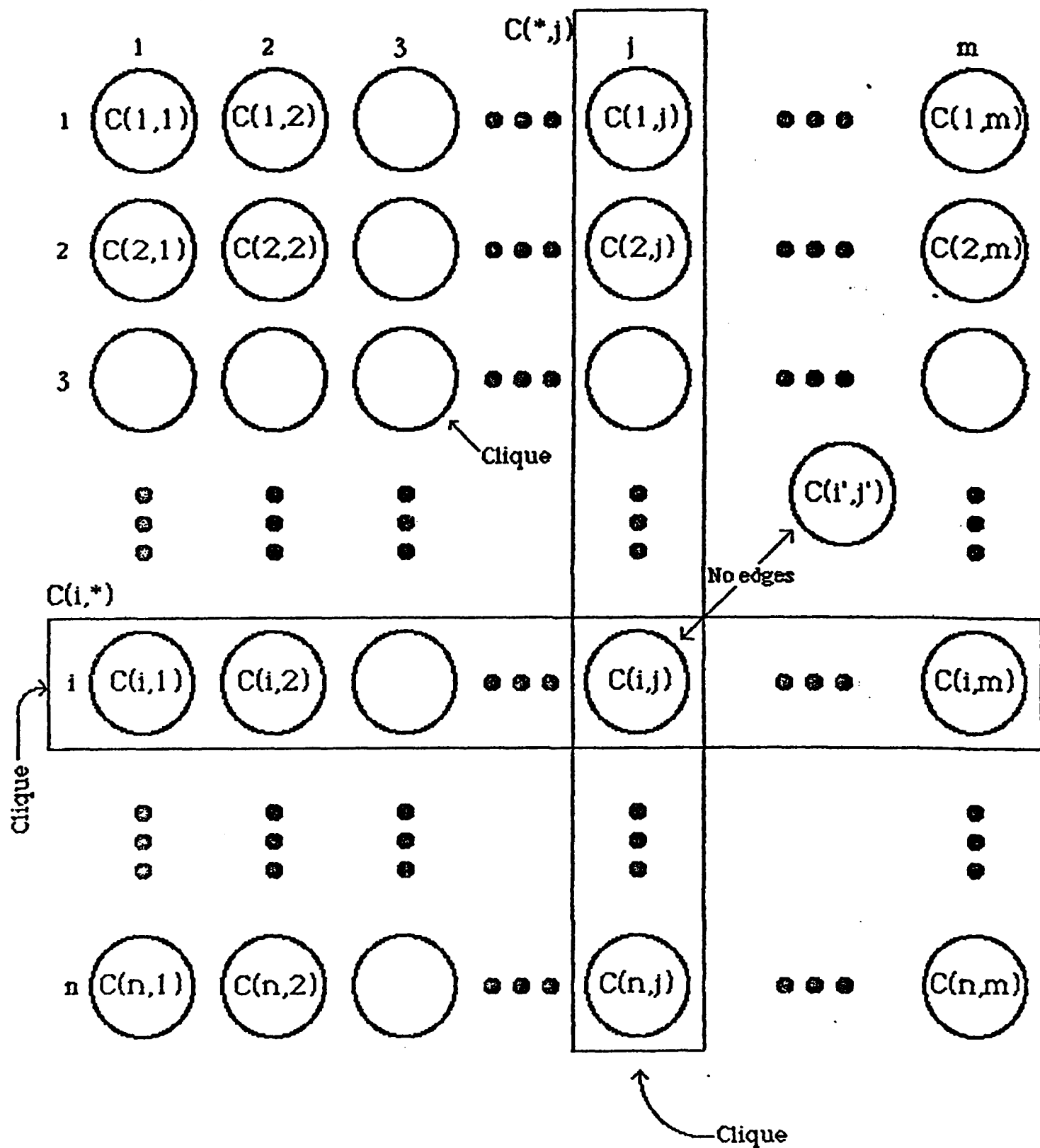


Figure 3.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER  ONR No. 84-1	2. GOVT ACCESSION NO.	3. RECIPIENT CATALOG NUMBER
4. TITLE  Irredundancy in multiple interval representation		5. TYPE OF REPORT & PERIOD COVERED  Technical Report
		6. PERFORMING ORGANIZATION REPORT NO.  Technical Report No. 411
7. AUTHOR(s)  Edward R. Scheinerman		8. CONTRACT OR GRANT NUMBER(s)  ONR No. N00014-79-C-0801
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematical Sciences The Johns Hopkins University Baltimore, Maryland 21218		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME & ADDRESS  Office of Naval Research Statistics and Probability Program Arlington, Virginia 22217		12. REPORT DATE  September, 1984
		13. NUMBER OF PAGES  19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS (of this report)  Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this report)  Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS  intersection graphs, redundancy		
20. ABSTRACT  It is shown that irredundant multiple interval intersection representations of a graph can be much more inefficient than representations which allow redundancies.		

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